Time and Frequency Crosstalk in Pulse-Modulated Systems

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The time and frequency crosstalk between Gaussian RF pulses sent via adjacent frequency channels over the same transmission medium is calculated. Shapes of the transfer characteristics of the transmitting and receiving filters vary from Gaussian to approximately that of a third-order maximally flat filter. The results permit one to design the transmitting and receiving transfer characteristics of adjacent PCM channels in such a way that the product of pulse spacing and channel spacing is minimized.

I. INTRODUCTION

Consider a transmission medium in which many simultaneous messages travel in one single direction. Each message, consisting of coded on-off RF pulses (PCM), has its own carrier and occupies a separate frequency channel. This occurs, for example, in the proposed long distance waveguide communication system.[1] The transmitter is considered as a filter through which the pulses of a message are fed to the transmission medium and the receiver as a filter that selectively couples the transmission medium to a detector.

The problem is to design these filters in such a way that the communication medium handles information at the highest possible rate. This means that the channels must be close to each other, providing high frequency occupancy, and that each message must be made of pulses close to each other, providing high time occupancy. In other words, we want to minimize the product of channel spacing and pulse spacing. What prevents us from making this product arbitrarily small is that, in general, a reduction of pulse and channel spacings implies an increase of time and frequency crosstalk, and these values are fixed by other considerations: the signal-to-noise level and the probability of errors allowed in the system. We shall see how they enter the picture.

The detector of each receiver reconstructs a message by deciding

whether or not a pulse is in the assigned time slot. For that purpose the detector operates only during sampling times that occur at the pulse repetition rate. Suppose that at a given sampling time there is no pulse to be detected; the detector nevertheless receives a signal which is the superposition of three types of interferences: trailing and leading edges of neighboring pulses, or time crosstalk; leakage from pulses in neighboring channels, or frequency crosstalk; and, of course, the main offender, thermal noise. If this signal is bigger than the slicing level the detector decides that there is a pulse in that time slot, and an error is made. Similarly, suppose that there is a pulse to be detected, but that superposed on it are time and frequency crosstalk and noise. If the total amplitude is smaller than the slicing level, the detector decides that a pulse does not exist in that time slot, and another error results.

Quantitative relations between the probability of errors of a system and the three interferences, thermal noise, time crosstalk, and frequency crosstalk were established in the companion paper.[2] The system considered there was such that time crosstalk came from the trailing edge of the pulse in the preceding time slot and from the leading edge of the pulse in the following time slot, while frequency crosstalk came from a single pulse of one of the neighboring channels.

The main result derived from that paper was that for a given probability of error there is a particular set of noise, time crosstalk, and frequency crosstalk levels that simultaneously minimizes time occupancy, frequency occupancy, and signal-to-noise ratio. In order to obtain this result, two conditions were imposed on the crosstalks:

(a) Each crosstalk must contribute with equal weight to the probability of errors; for that purpose, the time crosstalk per tail must be approximately 3 db below the frequency crosstalk.

(b) Time and frequency crosstalks must minimize by themselves the time and frequency occupancy of the system.

How do we design a system capable of satisfying both conditions? Our objective in this paper is to answer that question.

The variables in the system at our disposal to fulfill the required time and frequency crosstalks are:

shape, width, and time spacing of the input pulses;

sampling time;

synchronization of pulses of neighboring channels;

shape, width, and frequency spacing of the transfer characteristics of the sending and receiving filters;

transfer characteristic of the transmission medium.

These are too many variables to include in the problem simultaneously, so we assume that:

(a) The input pulses are Gaussian. This is not critical if the shape of the output pulse is determined essentially by the filtering characteristic of the system, as it should be.

(b) Sampling times in all the channels occur simultaneously. This type of synchronization is pessimistic because it introduces the maximum frequency crosstalk. The most favorable condition is obtained in general by maintaining synchronization but displacing by half a pulse spacing the messages, and consequently the sampling time, of every other one of the successive channels. If no synchronization among channels exists, the frequency crosstalk has a constant probability of acquiring any value between those of the two previous cases.

(c) The transfer characteristic of each filter is the normalized sum (maximum amplitude equal to one) of two Gaussian curves displaced from each other, as shown in Appendix A. Displacing the Gaussian curves yields a transfer characteristic which varies from that of a Gaussian filter to approximately that of a maximally flat filter with three resonant cavities, Fig. 1. Since the input signal is Gaussian and the transfer characteristic is a product of Gaussian functions, the mathematics involved in the calculations is simple. The filters have been idealized in the sense that they have linear phase characteristics. This introduces a constant delay between the input and output signal that is ignored altogether, since it only represents displacement of the time origin. The effect of small departures from linear phase can be evaluated using perturbation techniques.[3]

(d) The transfer characteristic of the transmission medium is unity. This implies that time crosstalk due to multipath transmission and to imperfect phase equalization is negligible compared to that derived from the filters. In a real system it may be desirable to have these two contributions to time crosstalk be of the same order of magnitude.

II. SYSTEM ANALYSIS

Definitions of symbols (see Fig. 2):

$2T$ = width of input Gaussian pulse measured at $1/e$ of the maximum amplitude (8.686 db down),

$1/\tau$ = pulse repetition frequency,

$2T_r$ = sampling time,

$2F_1$ = bandwidth of transmitting filter measured at half power,

$2F_2$ = bandwidth of receiving filter measured at half power,

a = parameter defining shape of transmitting filter, which can be selected from Gaussian to approximately maximally flat (three resonant branches, maximally flat filter),

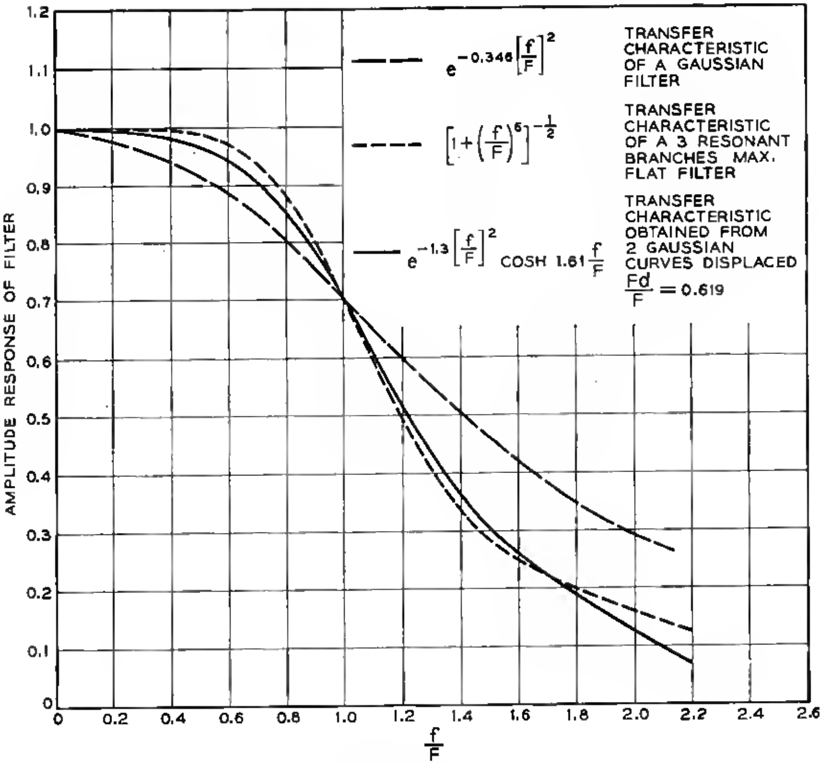

Fig. 1 — Transfer characteristics of different filters.

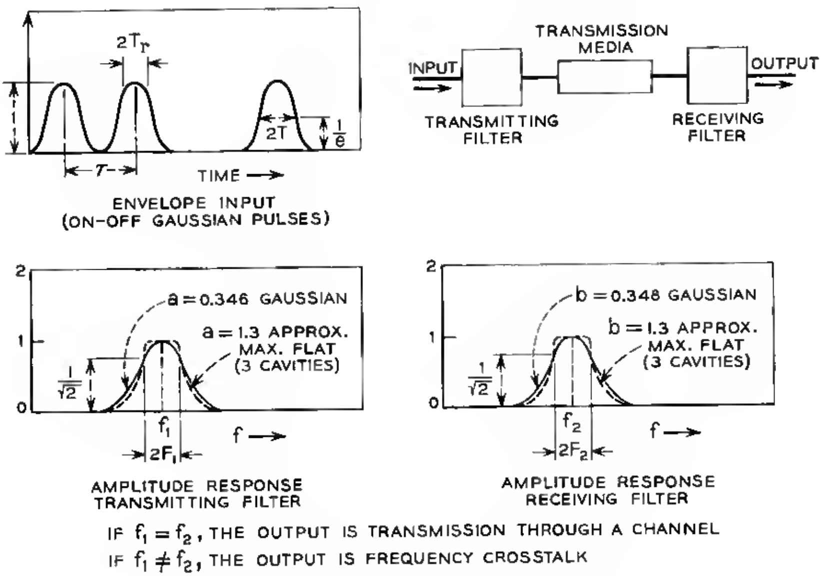

Fig. 2 — Definitions of symbols.

b = parameter defining shape of receiving filter,
f_1 = center frequency of transmitting filter,
f_2 = center frequency of receiving filter,
$\mu = F_1/F_2$ = ratio of transmitting to receiving bandwidth.
$\rho = |f_1 - f_2|/2F_2$ = ratio of channel spacing to the bandwidth of the receiving filter.

Finally, a useful parameter throughout our calculation is

$$k = \frac{4TF_1F_2}{\sqrt{F_1^2 + F_2^2}}.$$

If sending and receiving filters are Gaussian, k measures the pulse width $2T$ times the bandwidth of the system.

The envelope of the transient of a pulse through a channel and the maximum frequency crosstalk between two channels have been derived in Appendix B, equations (24) and (25). From them it is possible to calculate, in a way that will be described later, three functions that determine the best choice of transfer characteristics of the filters and channel spacing for a specified time and frequency crosstalk. Using the width of the input pulse $2T$ as a normalizing factor, those three functions are:

i. Normalized band spacing,

$$2T \mid f_1 - f_2 \mid = \frac{\tau \mid f_1 - f_2 \mid}{\theta + \dfrac{T_r}{2T}};$$

ii. Normalized receiver bandwidth,

$$4TF_2;$$

iii. Their ratio,

$$\rho = \frac{\mid f_1 - f_2 \mid}{2F_2};$$

where

$$\theta = \frac{\tau - T_r}{2T}.$$

In practical cases, the sampling time $2T_r$ is small compared to the pulse spacing τ, and θ becomes the normalized pulse spacing.

The three functions i, ii, and iii are plotted in Figs. 3, 4, and 5 for $\theta = 1.25$, and in Figs. 6, 7, and 8 for $\theta = 1.5$. They are derived as follows:

(a) We plot, (24), the transient of a pulse through a channel, and the

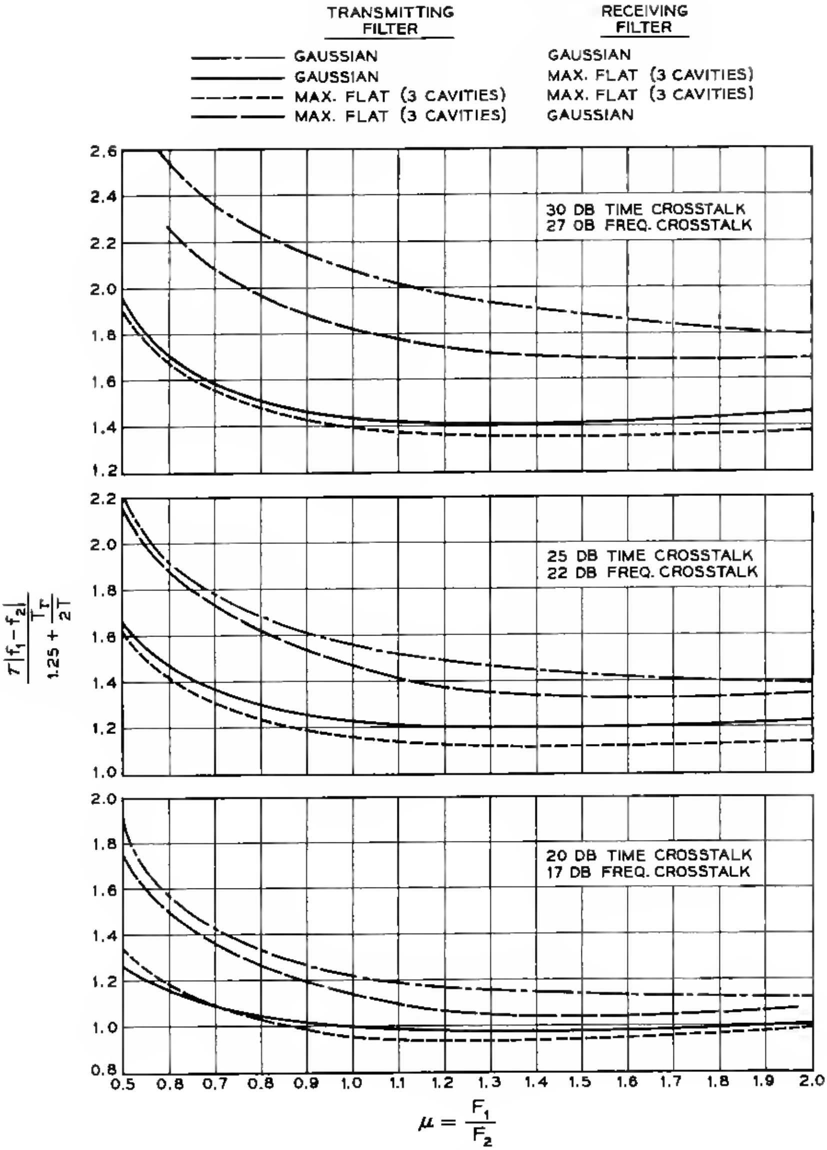

Fig. 3 — Normalized $\tau\,|f_1 - f_2|$ for $\theta = 1.25$.

maximum frequency crosstalk between neighboring channels, (25), for each possible combination of transfer characteristics of transmitting and receiving filters, and the ratio μ between sending and receiving bandwidths. Only one pair of these plots, Figs. 9 and 10, is included in this

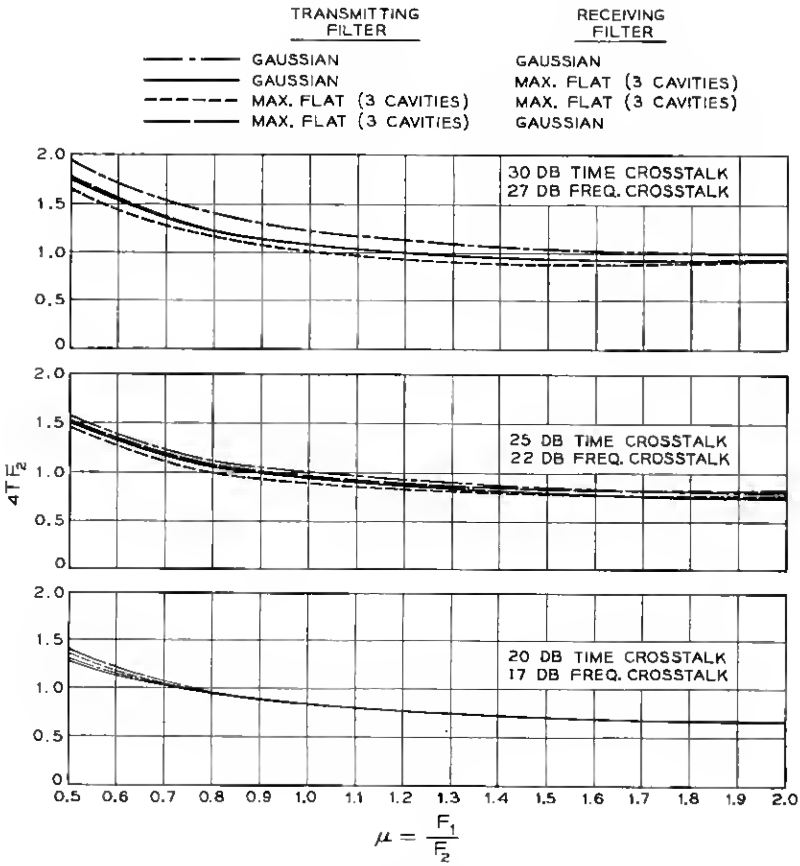

Fig. 4 — Normalized receiver bandwidth for $\theta = 1.25$.

paper to illustrate one example. The selected system has a Gaussian transmitting filter ($a = 0.346$, $m = 0$), approximately maximally flat receiving filter ($b = 1.6$, $n = 1.61$), and bandwidth ratio $\mu = 1$. Fig. 9 depicts the time response to a Gaussian pulse $2T$ wide at 8.686 db, through the two filters with variable over-all bandwidth of the system. The parameter

$$k = \frac{4TF_1}{\sqrt{1 + \mu^2}}$$

is proportional to that over-all bandwidth. Fig. 10 gives the maximum frequency crosstalk for the same Gaussian pulse through two filters

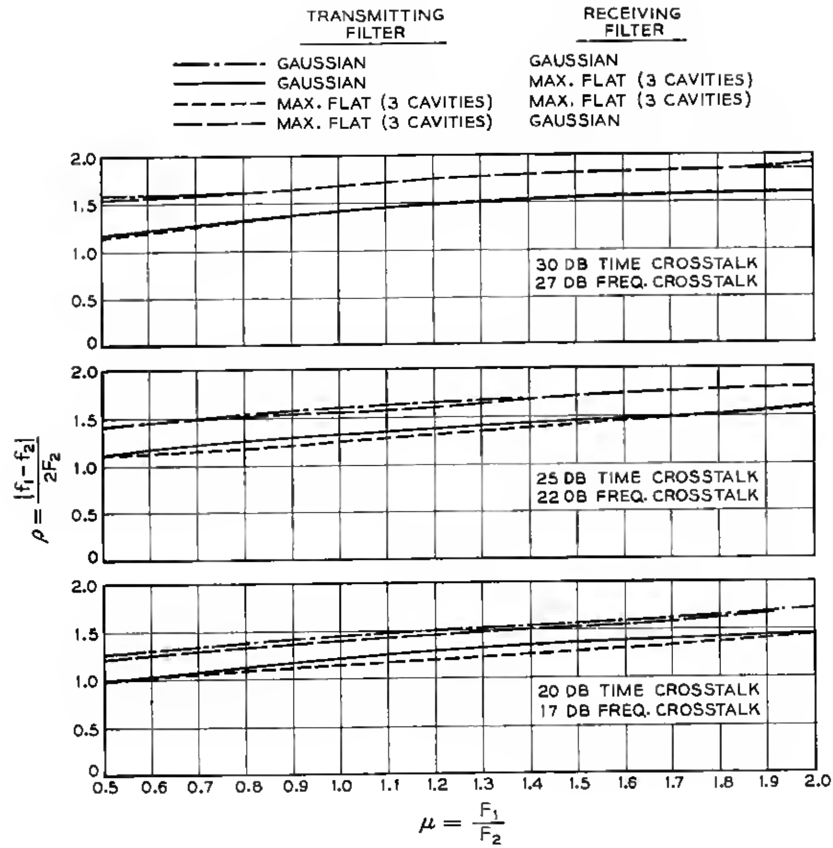

Fig. 5 — Ratio ρ for $\theta = 1.25$.

with fixed transfer characteristic shapes and bandwidth ratio. The abscissa

$$\rho = \frac{|f_1 - f_2|}{2F_2}$$

measures the channel spacing related to the receiver bandwidth, and the parameter is again k.

(b) From Fig. 9 we determine the smallest value of k, (smallest bandwidth of the system), compatible with given values of pulse spacing τ, sampling time $2T_1$, and allowed time crosstalk, by following these steps:

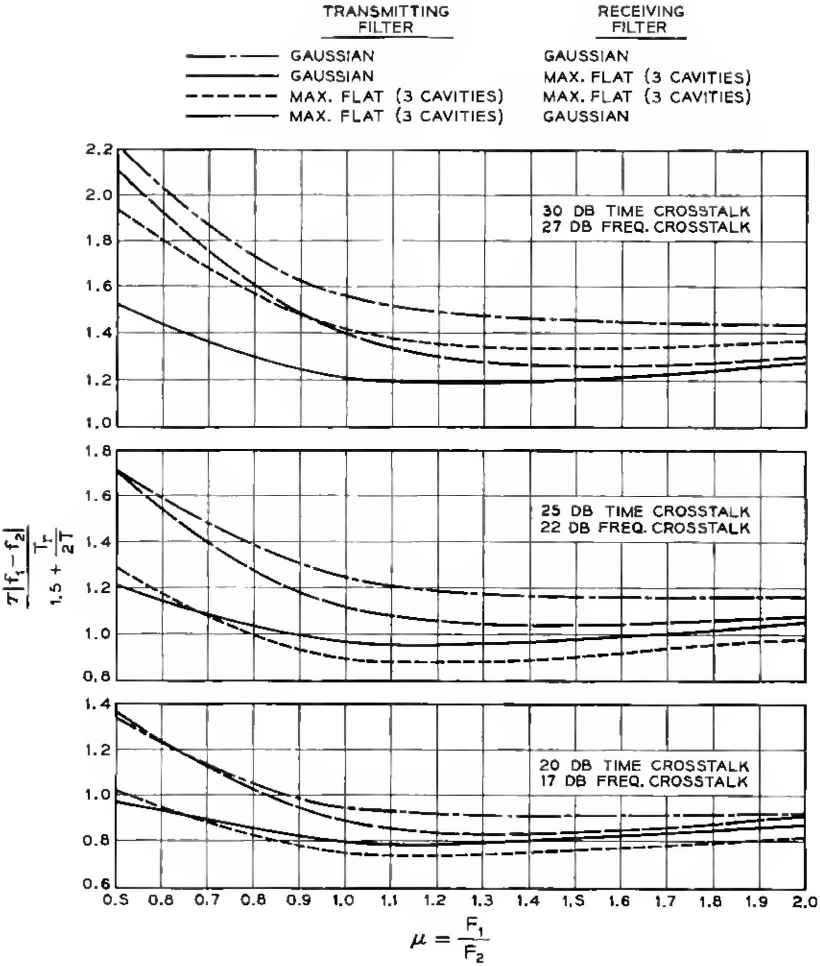

Fig. 6 — Normalized $\tau |f_1 - f_2|$ for $\theta = 1.5$.

1. Locate the normalized sampling time. This period, during which time crosstalk takes place, falls between the abscissa values

$$\frac{t}{2T} = \frac{\tau - T_r}{2T} \quad \text{and} \quad \frac{t}{2T} = \frac{\tau + T_r}{2T}.$$

2. Determine the ordinate that measures the allowed time crosstalk level (20-, 25-, and 30-db levels are indicated by dashed lines).

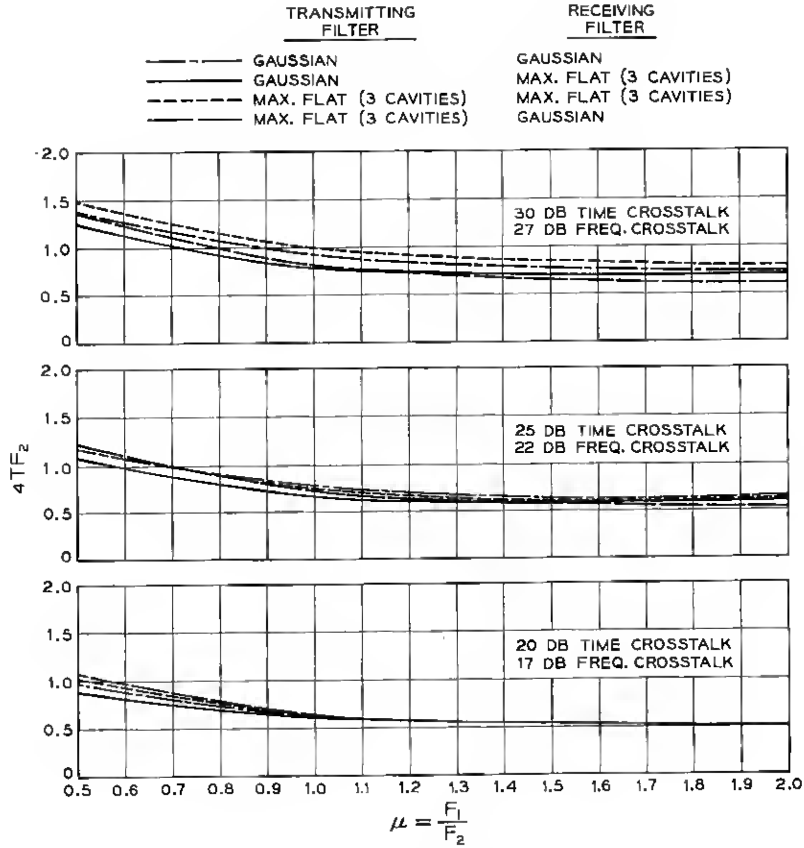

Fig. 7 — Normalized receiver bandwidth for $\theta = 1.5$.

3. Pick the curve with smallest k which, during the sampling time, is always below the allowed time crosstalk level.

(c) From Fig. 10 we determine the value of

$$\rho = \frac{|f_1 - f_2|}{2F_2}$$

by reading the abscissa of the point defined by the allowed frequency crosstalk ordinate (17-, 22-, and 27-db levels are indicated with dashed lines), and the curve characterized by the value of k deduced previously.

(d) The functions i, ii, and iii are derived after some arithmetic from μ, k, and ρ.

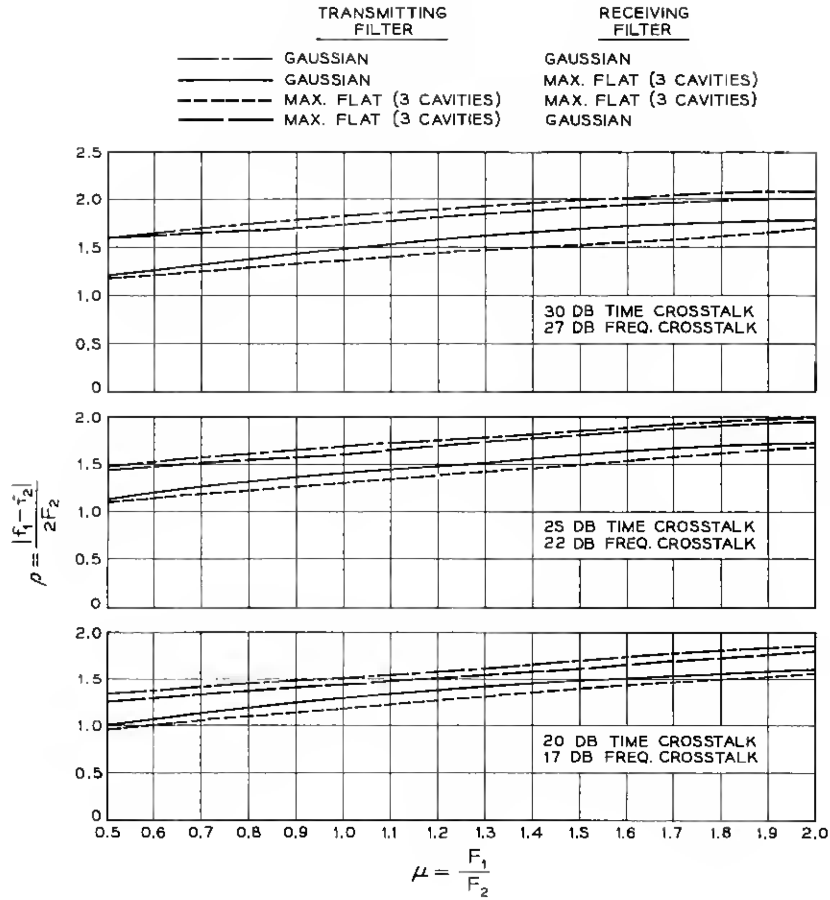

Fig. 8 — Ratio ρ for $\theta = 1.5$.

III. DISCUSSION OF RESULTS

Consider Fig. 3, in which the normalized pulse spacing

$$\frac{\tau}{2T} = 1.25 + \frac{T_r}{2T}$$

has been obtained assuming $\theta = 1.25$. The minimization of the product of pulse spacing times channel spacing $\tau \mid f_1 - f_2 \mid$ is achieved by making the sampling time T_r as short as possible, using maximally flat sending and receiving filters, and dividing the filtering in such a way that

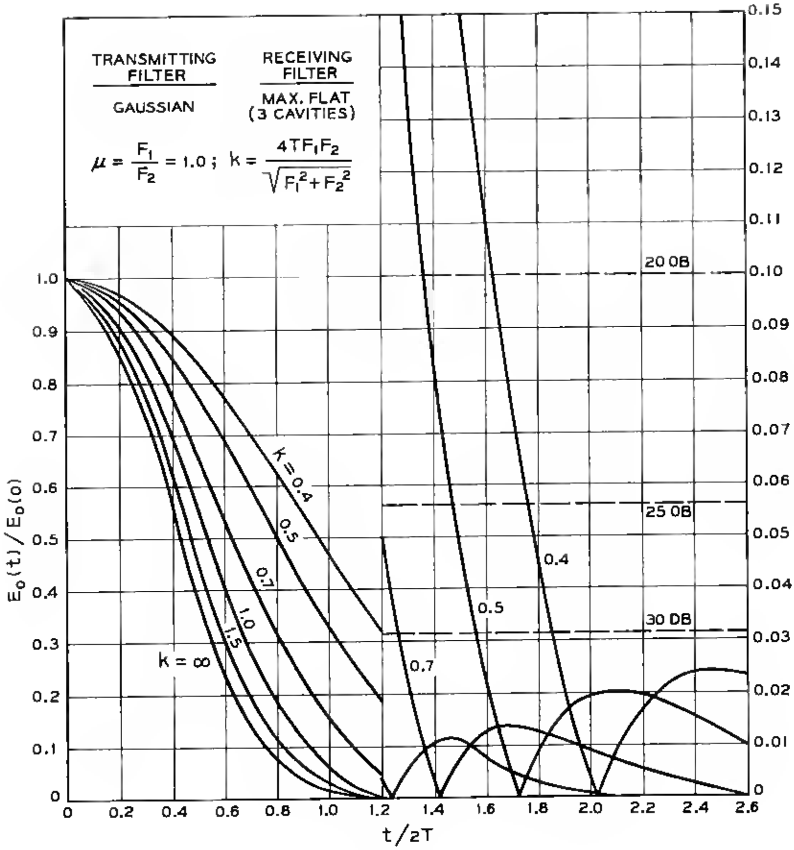

Fig. 9 — Typical plot of transient response to Gaussian input pulse.

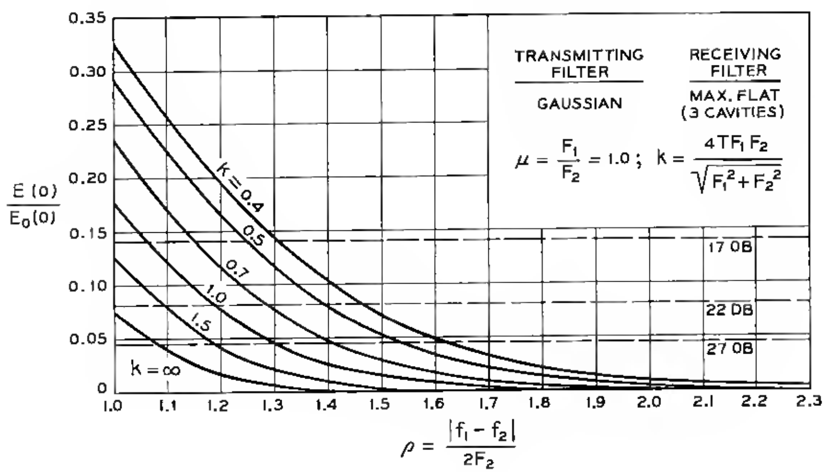

Fig. 10 — Typical plot of maximum crosstalk between neighboring channels.

$$\mu = \frac{F_1}{F_2} \cong 1.3.$$

The three sets of curves in Fig. 3 show that in each set the curves corresponding to maximally flat filters at the receiving end are very similar. Thus, as long as the receiving filter is maximally flat, there is no big advantage in using Gaussian or maximally flat filters at the transmitting end. Nevertheless, using shorter input pulses ($\theta = 1.5$), the normalized pulse spacing becomes

$$\frac{\tau}{2T} = 1.5 + \frac{T_r}{2T},$$

and it can be seen from Fig. 6 that, for low levels of interference (30 db time crosstalk and 27 db frequency crosstalk), the tails of the pulses become so important that there is a strong advantage in using a Gaussian filter at the sending end.

In order to reduce frequency crosstalk, each system should have filters with steep sides, and, in order to reduce time crosstalk, the transfer characteristics should have sloping sides. Figs. 3 and 6 analyzed previously, verify that a good compromise is obtained with a steep sided characteristic at the receiving end and a sloping one at the transmitting end.

Now, the minimums in the curves of Figs. 3 and 6 are very broad. We should select the ratio of bandwidths, μ, as large as possible, because the narrow band at the receiving end reduces the noise level. This must not be carried to extremes, because if μ is large enough the bandwidth of the sending filter may be broader than the channel spacing, and a pulse launched in one filter may waste a lot of power in a neighboring transmitting filter before reaching the receiver. This effect turns out to be of paramount importance when the transmitter characteristic is achieved by staggering filters at RF and IF; since, in this case, the RF filter may have even a wider band than that of the transmitter.

That problem, as well as the design of the receiving filter to have low noise level and the influence of the transmitter's peak power limitation on the filtering, will be discussed in another paper.[4]

We conclude with some design examples, using the following data:

(a) pulse repetition frequency $1/\tau = 160$ mc;

(b) allowed time crosstalk = 30 db;

(c) allowed frequency crosstalk = 27 db;

(d) if the pulses are narrow, say $\theta = 1.5$, then pulse width $2T$, pulse spacing τ, and sampling time $2T_r$ are related by the expression

$$2T = \frac{\tau - T_r}{1.5}.$$

Data (b), (c), and (d) locate the design curves as those in the upper group of Figs. 6, 7, and 8. The upper group of curves in Fig. 6 shows that the lowest value of the product of pulse spacing τ times channel spacing $|f_1 - f_2|$ (maximum rate of transmitted information) is the smallest minimum for the full line. This defines the shape of the filters as

transmitting filter: Gaussian,

receiving filter: approximately maximally flat (three cavities).

The abscissa and the ordinate of that minimum are

$$\mu = \frac{F_1}{F_2} = 1.3,$$

$$2T \mid f_1 - f_2 \mid = \frac{\tau \mid f_1 - f_2 \mid}{1.5 + \dfrac{T_r}{2T}} = 1.18.$$

In the upper-group curves of Figs. 7 and 8, the solid lines' ordinates corresponding to the abscissa $\mu = 1.3$ are

$$4TF_2 = 0.74,$$

$$\frac{\mid f_1 - f_2 \mid}{2F_2} = 1.6.$$

Solving the last three formulas for sampling times zero and half pulse width, we obtain

$2T_r$ (mμsec)	$2T$ (mμsec)	$2F_1$ (mc)	$2F_2$ (mc)	$\mid f_1 - f_2 \mid$ (mc)
0	4.16	231	178	284
1.78	3.57	269	207	331

The input pulse widths $2T$ are different because for the narrow pulses considered

$$2T = \frac{\tau - T_r}{1.5}$$

varies with the sampling time $2T_r$.

Now we shall see what happens when datum (d) of the previous example is changed from narrow pulses ($\theta = 1.5$) to broad pulses ($\theta = 1.25$). Then,

$$2T = \frac{6.25 - T_r}{1.25}$$

and the design answers are derived as in the previous example. The dashed lines of the upper-group curves in Figs. 3, 4, and 5 yield

transmitting filter: approximately maximally flat (three cavities),
receiving filter: approximately maximally flat (three cavities),

$$\mu = 1.4,$$

$$2T \mid f_1 - f_2 \mid = \frac{\tau \mid f_1 - f_2 \mid}{1.25 + \dfrac{T_r}{2T}} = 1.35,$$

$$4TF_2 = 0.9,$$

$$\frac{\mid f_1 - f_2 \mid}{2F_2} = 1.5.$$

Solving these equations for sampling time zero and half the input pulse width, we obtain

$2T_r$ (mμsec)	$2T$ (mμsec)	$2F_1$ (mc)	$2F_2$ (mc)	$\mid f_1 - f_2 \mid$ (mc)
0	5	252	180	270
2.08	4.16	302	216	325

The dotted and the solid lines in the upper group of curves of Figs. 3, 4, and 5 are very close to each other, and consequently there is no big advantage in using either a Gaussian or an approximately maximally flat filter (three cavities) at the transmitting end, while in the example of the narrow input pulse, the use of a transmitting Gaussian filter was definitely advantageous.

In the last table of results we notice that the transmitting filter bandwidth $2F_1$ is close to the channel spacing $\mid f_1 - f_2 \mid$, and consequently the power fed from one sending filter to a neighboring sending filter may be too large. To reduce this waste of power it is advisable to redesign the system, adopting a value of μ different from the "optimum," for example, $\mu = 1.1$. We shall see that because of the flatness of dashed line in Fig. 3, the increase in channel spacing is small.

Following the instructions of the first example,

$$2T \mid f_1 - f_2 \mid = \frac{\tau \mid f_1 - f_2 \mid}{1.25 + (T_r/2T)} = 1.37,$$

$$4TF_2 = 0.98,$$

$$\frac{\mid f_1 - f_2 \mid}{2F_2} = 1.4,$$

and

$2T_r$ (mμsec)	$2T$ (mμsec)	$2F_1$ (mc)	$2F_2$ (mc)	$\lvert f_1 - f_2 \rvert$ (mc)
0	5	216	196	274
2.08	4.16	260	236	329

Comparing this table of results with the previous one, we notice that for all sampling times the sending filter bandwidth has been substantially reduced, by approximately 16 per cent, at the expense of an increase in the receiving bandwidth of 9 per cent, and a very small increase of channel spacing of 1 per cent.

The bad influence of long sampling time can be appreciated by comparing the results in the first line in the table of the first example with the last line in the table of the last example. Both systems have equal input pulse widths of 4.16 millimicroseconds, but they have different sampling times, zero and 2.08 respectively. The differences between these two systems can be qualitatively justified by analyzing the necessary changes to pass from the first to the second. By increasing the sampling time, the time crosstalk increases, and, in order to maintain it at 30 db, the sending and receiving filters must be broadened 12 and 33 per cent, respectively. This bandwidth broadening increases the overlapping of transfer characteristics of neighboring channels, and therefore the frequency crosstalk goes up. To reduce it to the original level, 27 db, the channel spacing must increase 16 per cent.

IV. CONCLUSIONS

Considering only time and frequency crosstalk, Figs. 3 through 8 allow one to design the transmitting and receiving filters of adjacent PCM channels capable of minimizing the product of time occupancy and frequency occupancy.

In general, the transmitting and receiving filters should be Gaussian and approximately maximally flat (three cavities) respectively. The sloping side of the first filter contribute towards high pulse-repetition frequency, and the steep sides of the second filter contributes toward narrow channel spacing.

If the input pulses are broad, it is slightly advantageous to use approximately maximally flat filters (three cavities) in the transmitting end also. Naturally, sampling time should be as short as possible.

One set of typical results is for

pulse repetition frequency = 160 mc,
input Gaussian pulse width (at 8.686 db) = 4 mμsec,
sampling time = 1 mμsec,
Gaussian transmitting filter ~250 mc wide, at 3 db,
approximately maximally flat (three-cavity) receiving filter
~200 mc wide, at 3 db,
channel spacing = ~300 mc.

APPENDIX A

Summation of Two Displaced Gaussian Functions

The summation of two equal Gaussian curves $2F_G$ wide at 8.686 db and displaced F_d and $-F_d$ from the origin is

$$e^{-[(f-F_d)/F_G]^2} + e^{[(f+F_d)/F_G]^2}.$$

Normalizing the ordinate at $f = 0$ to unity, the summation becomes

$$Y = \frac{e^{-[(f-F_d)/F_G]^2} + e^{-[(f+F_d)/F_G]^2}}{2e^{-(F_d/F_G)^2}},$$

which can be rewritten,

$$Y = e^{-a(f/F)^2} \cosh mf/F, \tag{1}$$

where $\pm F$ are the values of f at which $Y = 1/\sqrt{2}$ (3 db),

$$a = \left(\frac{F}{F_G}\right)^2, \tag{2}$$

$$m = \frac{2F\,F_d}{F_G}. \tag{3}$$

Then a and m are related by the equation

$$\sqrt{2}\, e^{-a} \cosh m = 1. \tag{4}$$

From (1) and (4) it follows that once $2F$, the 3db width of the curve Y is given, the shape of it depends exclusively in the parameter a. For

$$a = 0.346 \tag{5}$$

the Gaussian function, plotted in Fig. 1 as

$$Y = e^{-0.346(f/F)^2}, \tag{6}$$

is obtained.

For the particular value

$$a = 1.3, \tag{7}$$

$$Y = e^{-1.3(f/F)^2} \cosh 1.61 \left(\frac{f}{F}\right). \tag{8}$$

This function is also plotted in Fig. 1, together with the amplitude response

$$\left[1 + \left(\frac{f}{F}\right)^6\right]^{-\frac{1}{2}}$$

of a third-order maximally flat filter,[5] $2F$ wide at 3 db and centered at $f = 0$. These two curves are very similar except for the argument $f/F > 2$ (ordinates below 20 db), and consequently filters with these transfer characteristics are interchangeable as long as the tails are not important.

APPENDIX B

Gaussian Pulse Through Two Filters

Assume a Gaussian RF pulse of duration $2T$ measured at 8.686 db, and carrier f_1,

$$i(t) = \frac{1}{\sqrt{\pi}T} e^{-(t/T)^2} \cos 2\pi f_1 t. \tag{9}$$

Its Fourier transform is

$$g(f) = \int_{-\infty}^{\infty} e^{-i2\pi ft} i(t)\, dt = \tfrac{1}{2}(e^{-[\pi T(f-f_1)]^2} + e^{-[\pi T(f+f_1)]^2}). \tag{10}$$

Passing this pulse through two filters with transfer frequency characteristics $Y_1(|f| - f_1)$ and $Y_2(|f| - f_2)$, the output signal is

$$e(t) = \int_{-\infty}^{\infty} e^{i2\pi ft} Y_1(|f| - f_1) Y_2(|f| - f_2) g(f)\, df. \tag{11}$$

Substituting $g(f)$ from (10) into (11) and changing variables, one obtains

$$\begin{aligned} e(t) = {} & \frac{e^{i2\pi f_1 t}}{2} \int_{-\infty}^{\infty} Y_1(|f + f_1| - f_1) Y_2(|f + f_1| - f_2) \\ & \qquad \cdot e^{-(\pi f T)^2 + i2\pi ft}\, df \\ & + \frac{e^{-i2\pi f_1 t}}{2} \int_{-\infty}^{\infty} Y_1(|f - f_1| - f_1) Y_2(|f - f_1| - f_2) \\ & \qquad \cdot e^{-(\pi f T)^2 + i2\pi ft}\, df, \end{aligned} \tag{12}$$

and since the terms are complex conjugate,

$$e(t) = \operatorname{Re} e^{i2\pi f_1 t} \int_{-\infty}^{\infty} Y_1(\,|f + f_1| - f_1) \cdot Y_2(\,|f + f_1| - f_2) e^{-(\pi f T)^2 + i2\pi f t}\, df. \tag{13}$$

The envelope is

$$E(t) = \left| \int_{-f_1}^{\infty} Y_1(f) Y_2(f + f_1 - f_2) e^{-(\pi f T)^2 + i2\pi f t}\, df + \int_{-\infty}^{-f_1} Y_1(-f - 2f_1) Y_2(-f - f_1 - f_2) e^{-(\pi f T)^2 + i2\pi f t}\, df \right|. \tag{14}$$

Since there are many RF cycles in the pulse $f_1 T \gg 1$, the second integral is negligible and

$$E(t) \cong \left| \int_{-\infty}^{\infty} Y_1(f) Y_2(f + f_1 - f_2) e^{-(\pi f T)^2 + i2\pi f t}\, df \right|. \tag{15}$$

Furthermore, if the transfer characteristics of the filters are displaced Gaussians, (1), the envelope of the output pulse becomes

$$E(t) = \left| \int_{-\infty}^{\infty} e^{-a(f/F_1)^2 - b[(f+f_1-f_2)/F_2]^2 - (\pi f T)^2 + i2\pi f t} \cosh \frac{mf}{F_1} \cdot \cosh n \left(\frac{f + f_1 - f_2}{F_2} \right) df \right|, \tag{16}$$

which can be normalized to $E_0(0)$, the output at the instant $t = 0$ through two filters centered at the same frequency, $f_1 = f_2$. Then,

$$\frac{E(t)}{E_0(0)} = \frac{\left| \int_{-\infty}^{\infty} e^{-a(f/F_1)^2 - b[(f+f_1-f_2)/F_2]^2 - (\pi f T)^2 + i2\pi f t} \cosh \frac{mf}{F_1} \cosh n \left(\frac{f + f_1 - f_2}{F_2} \right) df \right|}{\int_{-\infty}^{\infty} e^{-a(f/F_1)^2 - b(f/F_2)^2 - (\pi f T)^2} \cosh \frac{mf}{F_1} \cosh \frac{nf}{F_2}\, df}. \tag{17}$$

Performing the integrations,

$$\begin{aligned} \frac{E(t)}{E_0(0)} = \frac{e^{-4\rho^2 bB(1+aA^2) - B(t/T)^2}}{4 \cosh \frac{1}{2} mn\mu A^2 B} &\cdot \Big| e^{B[2n\rho(1+aA^2) - 2mb\rho A^2 + i(m+n\mu)A(t/T) + mn\mu A^2/2]} \\ &+ e^{B[-2n\rho(1+aA^2) - 2mb\rho A^2 + i(m-n\mu)A(t/T) - mn\mu A^2/2]} \\ &+ e^{B[-2n\rho(1+aA^2) + 2mb\rho A^2 - i(m+n\mu)A(t/T) + mn\mu A^2/2]} \\ &+ e^{B[+2n\rho(1+aA^2) + 2mb\rho A^2 - i(m-n\mu)A(t/T) - mn\mu A^2/2]} \Big|, \end{aligned} \tag{18}$$

in which,

$$\rho = \frac{|f_1 - f_2|}{2F_2}, \tag{19}$$

$$\mu = \frac{F_1}{F_2}, \tag{20}$$

$$A = \frac{1}{\pi F_1 T} = \frac{4}{\pi k \sqrt{1 + \mu^2}}, \tag{21}$$

$$k = \frac{4TF_1F_2}{\sqrt{F_1^2 + F_2^2}}, \tag{22}$$

$$B = \frac{1}{1 + A^2(a + b\mu^2)}. \tag{23}$$

If both filters are centered at the same frequency, $\rho = 0$, the normalized output pulse (18) becomes

$$\begin{aligned}\frac{E_0(t)}{E_0(0)} &= \frac{e^{-B(t/T)^2}}{2 \cosh \frac{1}{2} mn\mu A^2 B} \\ &\cdot \left[e^{\frac{1}{2} mn\mu A^2 B} \cos\, (m + n\mu) AB\, \frac{t}{T} + e^{\frac{1}{2} mn\mu A^2 B} \cos\, (m - n\mu) AB\, \frac{t}{T} \right].\end{aligned} \tag{24}$$

It also follows from (17) that the maximum amplitude transmitted through the two filters centered at different frequencies occurs at $t = 0$, and that its normalized value at $t = 0$ derived from (18) is

$$\begin{aligned}\frac{E(0)}{E_0(0)} &= \frac{e^{-4\rho^2 bB(1+aA^2)}}{2 \cosh \frac{1}{2} mn\mu A^2 B} \\ &\cdot \{ e^{\frac{1}{2} mn\mu A^2 B} \cosh 2\rho B[n + A^2(na - mb)] + e^{-\frac{1}{2} mn\mu A^2 B} \\ &\cdot \cosh 2\rho B[n + A^2(na + mb)] \}.\end{aligned} \tag{25}$$

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